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ASYMPTOTIC PROPERTIES OF STOCHASTIC GREEDY BIN-PACKING

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Abstract

An adaptive or greedy policy for packing I bins, or equivalently for scheduling jobs for the attention of a number, I, of processors is studied. It is shown that the suitably normalized bin contents become nearly jointly but degenerately Gaussian/normal if the rate of approach of jobs becomes large. Explicit and simple parameter characterizations are supplied and the asymptotics are compared with simulation. The advantage of the greedy policy over a *laissez-faire* policy of equal access is quantified, and seen to depend upon $\sqrt{\text{number of bins or processors}}$.

Introduction

We study an adaptive or greedy policy for packing I bins, or equivalently for scheduling jobs for the attention of a number, I, of processors. The connection between bin-packing and makespan scheduling is well described in Coffman and Lueker [1], chapter 1.

It is shown that the suitably normalized bin contents become nearly jointly but degenerately Gaussian/normal if the rate of approach of jobs becomes large. Explicit and simple parameter characterizations are supplied, and the asymptotics are compared with simulation. The advantage of the greedy policy over a *laissez-faire* policy of equal access is quantified, and seen to depend upon $\sqrt{\text{number of bins or processors}}$.

Greedy Bin-Filling

There are I bins (potential servers). Jobs arrive at the bin system according to a homogeneous Poisson process, rate λ. Each job size is independently and identically distributed according to $F_B(x)$; a generic job size is B, a real positive random variable. Let $N_i(t)$ denote *amount of work* in bin i at time t; this is the sum of the job sizes deposited in that bin up to time t. We refer to $N_i(t)$ as the *bin size* at time t hereafter, although other terminology is used in the scheduling literature.

Consider *Policy* E (Equalization): When a new job arrives it is deposited in the bin with smallest amount of accumulated work. $N(t) = \{N_i(t),\ t \geq 0,\ i \in (1, 2, \ldots, I)\}$ is a Markov process with the following generator; for $\Delta > 0$

$$
\begin{aligned}
N_i(t+\Delta) &= N_i(t) && \text{with probability } 1 - \lambda\Delta p_i(N_i(t), N(t)) + o(\Delta) \\
&= N_i(t) + B(t) && \text{with probability } \lambda\Delta p_i(N_i(t), N(t)) + o(\Delta) \qquad (1)
\end{aligned}
$$

Define

$$
p_i(N_i(t), N(t)) = \begin{cases} 1 & \text{if } N_i(t) \leq N_j(t), j \neq i \\ 0 & \text{otherwise} \end{cases}
$$

Consider also *Probabilistic Policy* E: same as above but

$$
p_i(N_i(t), N(t)) = \frac{h(N_i(t))}{\sum_{j=1}^{I} h(N_j(t))} \qquad (2)
$$

where $h(\cdot)$ is a positive homogeneous function chosen to become large when $N_i(t)$ is small; e.g. $h(x) = x^{-p}$. Such a policy leads to workload growth very similar to Policy E's, but can be analyzed more readily. See Gaver, Morrison and Silveira [2] for application of such a probabilistic policy in a service-system scheduling context. In the present context, the bins can be buffers containing jobs to be processed later and the aim is to keep the total processing time short; it is (optimistically) assumed that the contents of each bin is known at all times to the scheduler and that each processing time is known when the job or task appears.

Joint Moment-Generating Function

Let the moment-generating function (assumed to exist, otherwise use the characteristic function) be

$$\psi(\theta,t)\overset{(d)}{=}E\left[e^{\theta\cdot N(t)}\right]\equiv E\left[\exp\left(\sum_{j=1}^{I}\theta_j N_j(t)\right)\right]. \tag{3}$$

Condition on $N_i(t)$, $i\ \varepsilon\ (1, 2, \ldots, I)$ and use the generator to obtain

$$E\left[\exp\left(\sum_{j=1}^{I}\theta_j N_j(t+\Delta)\right)\middle| N(t),B(t)\right]$$

$$=[1-\lambda\Delta]\exp\left(\sum_{j=1}^{I}\theta_j N_j(t)\right)+$$

$$+\lambda\Delta\sum_{j=1}^{I}\left[\exp\left(\theta_j\left(N_j(t)+B(t)\right)+\sum_{k\neq j}\theta_k N_k(t)\right)p_j\left(N_j(t);N(t)\right)\right]+o(\Delta).$$

Remove conditions, defining the m g f of the task size arriving at t, $B(t)$, to be $\hat{b}(\theta)$, to find

$$\psi(\theta,t+\Delta)=(1-\lambda\Delta)\psi(\theta,t)+\lambda\Delta\sum_{j=1}^{I}\hat{b}(\theta_j)E\left[\exp\left(\theta_j N_j(t)\right)\cdot p_j\left(N_j(t);N(t)\right)\right]+o(\Delta).$$

Let $\Delta\rightarrow 0$ to get

$$\frac{\partial\psi(\theta,t)}{\partial t}=-\lambda\psi(\theta,t)+\lambda\sum_{j=1}^{I}\hat{b}(\theta_j)E\left[\exp\left(\theta_j N_j(t)\right)\cdot p_j\left(N_j(t);N(t)\right)\right]. \tag{4}$$

Note that nothing that follows prevents non-stationary input rates: i.g. $\lambda=\lambda r(t)$, and the mgf of $B(t)$ to be $\hat{b}(\theta;t)$ provided these do not drop quickly to zero.

Scaling

Put

$$\mathbf{X}_j(t) = \frac{N_j(t) - \lambda\beta_j(t)}{\sqrt{\lambda}} \tag{5}$$

and let $\lambda \gg 1$. It is anticipated that with suitable choice of the functions $\{\beta_j(t)\}$ $\{\mathbf{X}_j(t),\ j \in (1,2,\ldots,I)\}$ should become a Gaussian process as $\lambda \to \infty$. Let $\varphi(\theta,t;\lambda) \overset{(d)}{=} E\left[\exp\left(\sum_{j=1}^{I}\theta_j\mathbf{X}_j(t)\right)\right]$ for $\mathbf{X}_j(t)$ defined as above. Note that

$$E\left[\exp\left(\sum_{j=1}^{I}\left(\theta_j/\sqrt{\lambda}\right)N_j(t)\right)\right] = \psi\left(\theta/\sqrt{\lambda},t\right)$$

$$= E\left[\exp\left(\sum_{j=1}^{I}\theta_j\mathbf{X}_j(t)\right)\right]\cdot\exp\left(\sqrt{\lambda}\sum_{j=1}^{I}\theta_j\beta_j(t)\right) \tag{6}$$

$$= \varphi(\theta,t;\lambda)\exp\left(\sqrt{\lambda}\,\theta\beta(t)\right).$$

Now since from (4)

$$\frac{\partial\psi\left(\theta/\sqrt{\lambda},t\right)}{\partial t} = -\lambda\psi\left(\theta/\sqrt{\lambda},t\right) + \lambda\sum_{j=1}^{I}\hat{b}\left(\theta_j/\sqrt{\lambda}\right)E\left[e^{\theta_j/\sqrt{\lambda}N_j(t)}\cdot p_j\left(N_j(t);N(t)\right)\right],$$

substitution of (5) yields

$$\frac{\partial\psi\left(\theta/\sqrt{\lambda},t\right)}{\partial t} = \frac{\partial}{\partial t}\left[\varphi(\theta,t;\lambda)\exp\left(\sqrt{\lambda}\,\theta\beta(t)\right)\right] = \frac{\partial\varphi(\theta,t;\lambda)}{\partial t}\exp\left(\sqrt{\lambda}\,\theta\beta(t)\right)$$

$$+\varphi(\theta,t;\lambda)\left(\sqrt{\lambda}\,\theta\beta'(t)\exp\left(\sqrt{\lambda}\,\theta\beta(t)\right)\right)$$

$$= -\lambda\varphi(\theta,t;\lambda)\exp\left(\sqrt{\lambda}\,\theta\beta(t)\right) +$$

$$+\lambda\sum_{j=1}^{I}\hat{b}\left(\theta_j/\sqrt{\lambda}\right)E\left[\exp(\theta\mathbf{X}(t))\cdot\exp\left(\sqrt{\lambda}\,\theta\beta(t)\right)\cdot p_j\left(\lambda\beta_j(t)+\sqrt{\lambda}\mathbf{X}_j(t);\lambda\beta+\sqrt{\lambda}\mathbf{X}\right)\right].$$

Cancel $\exp\left(\sqrt{\lambda}\,\theta\beta(t)\right)$ to obtain

$$\frac{\partial\varphi(\theta,t;\lambda)}{\partial t}+\varphi(\theta,t;\lambda)\sqrt{\lambda}\,\theta\beta'(t)=-\lambda\varphi(\theta,t;\lambda)$$

$$+\lambda\sum_{j=1}^{I}\hat{b}\left(\theta_j/\sqrt{\lambda}\right)E\left[\exp(\theta\mathbf{X}(t))\cdot p_j\left(\lambda\beta_j(t)+\sqrt{\lambda}\mathbf{X}_j(t);\lambda\beta+\sqrt{\lambda}\mathbf{X}\right)\right]. \tag{7}$$

Let

$$p_j\left(N_j(t),N(t)\right)=\frac{h\left(N_j(t)\right)}{\sum_{k=1}^{I}h_k(N_k(t))}=\frac{h\left(\lambda\beta_j(t)+\sqrt{\lambda}\mathbf{X}_j(t)\right)}{\sum_{k=1}^{I}h\left(\lambda\beta_k(t)+\sqrt{\lambda}\mathbf{X}_k(t)\right)}$$

where $h(\cdot)$ is homogeneous: $h(\lambda x)=\lambda^p h(x)$. Consequently

$$p_j\left(\lambda\beta_j(t)+\sqrt{\lambda}\mathbf{X}_j(t);\lambda\beta(t)+\sqrt{\lambda}\mathbf{X}(t)\right)=\frac{h\left(\beta_j(t)+\mathbf{X}_j(t)/\sqrt{\lambda}\right)}{\sum_{k=1}^{I}h\left(\beta_k(t)+\mathbf{X}_k(t)/\sqrt{\lambda}\right)} \tag{8}$$

($h(x)$ is assumed to be differentiable). Expand in inverse powers of $\sqrt{\lambda}$:

$$p_j\left(\lambda\beta_j(t)+\sqrt{\lambda}\mathbf{X}_j(t);\lambda\beta(t)+\sqrt{\lambda}\mathbf{X}(t)\right)=$$

$$\frac{h\left(\beta_j(t)\right)}{\sum_{k=1}^{I}h(\beta_k(t))}+\frac{1}{\sqrt{\lambda}}\left[\frac{h'\left(\beta_j(t)\right)\mathbf{X}_j(t)}{\sum_{k=1}^{I}h(\beta_k(t))}-\frac{h\left(\beta_j(t)\right)}{\left(\sum_{k=1}^{I}h(\beta_k(t))\right)^2}\sum_{k=1}^{I}h'(\beta_k(t))\mathbf{X}_k(t)\right]+O\left(\frac{1}{\lambda}\right). \tag{9}$$

Since

$$\sum_{j=1}^{I}p_j\left(\lambda\beta_j(t)+\sqrt{\lambda}\mathbf{X}_j(t);\lambda\beta(t)+\sqrt{\lambda}\mathbf{X}(t)\right)=1$$

$$=\sum_{j=1}^{I}\frac{h\left(\beta_j(t)\right)}{\sum_{k=1}^{I}h(\beta_k(t))} \tag{9,a}$$

this implies that the summed coefficients of $1/\sqrt{\lambda}, 1/\lambda$, *etc.* must be individually zero; this is easily verified for the coefficient of $1/\sqrt{\lambda}$.

Asymptotic Expansion For $\varphi(\theta,t;\lambda)$.

Put

$$\varphi(\theta,t;\lambda)=\varphi_0(\theta,t)+\frac{1}{\sqrt{\lambda}}\varphi_1(\theta,t)+\frac{1}{\lambda}\varphi_2(\theta,t)+\dots$$

This can now be entered into (7) and evaluated by means of (9):

$$\begin{aligned}
&\sum_{\ell=0}^{\infty}\frac{\partial\varphi_\ell}{\partial t}\cdot\frac{1}{\left(\sqrt{\lambda}\right)^\ell}+\sqrt{\lambda}\,\theta\beta'(t)\sum_{\ell=0}^{\infty}\varphi_\ell\cdot\frac{1}{\left(\sqrt{\lambda}\right)^\ell}=-\lambda\sum_{\ell=0}^{\infty}\varphi_\ell\frac{1}{\left(\sqrt{\lambda}\right)^\ell}\\
&+\lambda\sum_{j=1}^{I}\left(\sum_{k=0}^{\infty}b_k\frac{1}{k!}\theta_j^k/\left(\sqrt{\lambda}\right)^k\right)\Bigg[\left(\sum_{\ell=0}^{\infty}\varphi_\ell\frac{1}{\left(\sqrt{\lambda}\right)^\ell}\right)\left(\frac{h(\beta_j(t))}{\sum_{k=1}^{I}h(\beta_k(t))}\right)\\
&+\frac{1}{\sqrt{\lambda}}\left(\frac{h'(\beta_j(t))}{\sum_{k=1}^{I}h(\beta_k(t))}\left(\sum_{\ell=0}^{\infty}\frac{\partial\varphi_\ell}{\partial\theta_j}\frac{1}{\left(\sqrt{\lambda}\right)^\ell}\right)-\frac{h(\beta_j(t))}{\left(\sum_{k=1}^{I}h(\beta_k(t))\right)^2}\sum_{k=1}^{I}h'(\beta_k(t))\sum_{\ell=0}^{\infty}\frac{\partial\varphi_\ell}{\partial\theta_k}\frac{1}{\left(\sqrt{\lambda}\right)^\ell}\right)+O\left(\frac{1}{\lambda}\right)\Bigg].
\end{aligned}\tag{10}$$

In the above $b_k=E\left[B^k\right]$, the k^{th} moment of job size; of course $b_0=1$.

Now identify the coefficients of inverse powers of $\sqrt{\lambda}$, and thereby equations for φ_ℓ and β_ℓ. From (10) for $\ell=0$,

$$\frac{\partial \varphi_0}{\partial t} + \sqrt{\lambda}\,\theta\beta'(t)\cdot\varphi_0 = -\lambda\varphi_0 + \lambda\sum_{j=1}^{I}\left(1 + b_1\theta_j\frac{1}{\sqrt{\lambda}} + \frac{1}{2}b_2\theta_j^2\frac{1}{\lambda} + \ldots\right)$$

$$\left[\varphi_0\cdot\frac{h(\beta_j(t))}{\sum_{k=1}^{I}h(\beta_k(t))} + \frac{1}{\sqrt{\lambda}}\left(\frac{h'(\beta_j(t))}{\sum_{k=1}^{I}h(\beta_k(t))}\frac{\partial\varphi_0}{\partial\theta_j} - \frac{h(\beta_j(t))}{\left[\sum_{k=1}^{I}h(\beta_k(t))\right]^2}\sum_{k=1}^{I}h'(\beta_k(t))\frac{\partial\varphi_0}{\partial\theta_k}\right) + O\left(\frac{1}{\lambda}\right)\right]. \tag{11}$$

The terms of order λ cancel from the r h s. The terms of order $\sqrt{\lambda}$ on l h s and r h s cancel if

$$\theta\beta'(t)\varphi_0 = b_1\sum_{j=1}^{I}\theta_j\frac{h(\beta_j(t))}{\sum_k h(\beta_k(t))}\cdot\varphi_0.$$

In order for this to occur,

$$\frac{d\beta_j}{dt} = b_1\frac{h(\beta_j(t))}{\sum_k h(\beta_k(t))},\quad j = 1,2,\ldots,I \tag{12}$$

the solution of which determines $\beta_j(t)$.

Next look for terms of order 1. The l h s provides $\frac{\partial\varphi_0}{\partial t}$. The r h s provides the terms $\varphi_0\sum_{j=1}^{I}b_2\frac{\theta_j^2}{2}\frac{h(\beta_j(t))}{\sum_k h(\beta_k(t))}$

and $b_1\sum_{j=1}^{I}\theta_j\left(\frac{h'(\beta_j(t))}{\sum_k h(\beta_k(t))}\frac{\partial\varphi_0}{\partial\theta_j} - \frac{h(\beta_j(t))}{\left(\sum_k h(\beta_k(t))\right)^2}\sum_{k=1}^{I}h'(\beta_k(t))\frac{\partial\varphi_0}{\partial\theta_k}\right).$

Note that the condition (12) actually annihilates any term of order 1 (or higher) in φ_ℓ for $\ell = 1, 2, \ldots$ on the r h s, and the discussion of (9, a) shows that there is no contribution from $\partial\varphi_\ell/\partial\theta_j$. Consequently

$$\frac{\partial\varphi_0}{\partial t}=\frac{b_2}{2}\sum_{j=1}^{I}\theta_j^2 p_j\big(\beta_j(t)\big)\varphi_0(\theta,t)+$$

$$b_1\left[\sum_{j=1}^{I}\theta_j\left(\frac{h'\big(\beta_j(t)\big)}{\sum_k h(\beta_k(t))}\frac{\partial\varphi_0(\theta,t)}{\partial\theta_j}-\frac{h\big(\beta_j(t)\big)}{\left(\sum_k h(\beta_k(t))\right)^2}\sum_k h'(\beta_k(t))\frac{\partial\varphi_0(\theta,t)}{\partial\theta_k}\right)\right] \quad (13)$$

$$\equiv\frac{b_2}{2}\sum_{j=1}^{I}\theta_j^2 p_j\big(\beta_j(t)\big)\varphi_0(\theta,t)+b_1\sum_{j=1}^{I}\theta_j\left(H_j(t)\frac{\partial\varphi_0}{\partial\theta_j}-\sum_k H_{jk}(t)\frac{\partial\varphi_0(\theta,t)}{\partial\theta_k}\right).$$

Note: $\sum_{j=1}^{I}\left(H_j(t)\frac{\partial\varphi_\ell}{\partial\theta_j}\frac{1}{\left(\sqrt{\lambda}\right)^\ell}-\sum_k H_{jk}(t)\frac{\partial\varphi_\ell}{\partial\theta_k}\frac{1}{\left(\sqrt{\lambda}\right)^k}\right)=0.$

The PDE for $\varphi_0(\theta,t)$ is recognizable as that of an Ornstein-Uhlenbeck (Gaussian) process. Similar equations for higher-order corrections can be derived similarly, but we omit this step.

MOMENTS

The (0^{th} order) joint moments of $X(t)=(X_j(t), j=1,2,\ldots,I)$ satisfy ordinary differential equations that are readily obtained by differentiation of (13) at $\theta=\underline{0}$. If $V_i(t)=E\left[X_i^2(t)\right]$, $V_{ij}(t)=E\left[X_i(t)X_j(t)\right]$ for $i\neq j$ then

$$\frac{dV_i}{dt}=b_2p_i(\beta_i(t))+2b_1\left[H_i(t)V_i(t)-\sum_{(k)}H_{ik}(t)V_{ik}(t)\right]$$

$$\frac{dV_{ij}}{dt}=b_1\left[\big(H_i(t)+H_j(t)\big)V_{ij}(t)-\left(\sum_{(k)}H_{jk}(t)V_{ik}(t)+\sum_{(k)}H_{ik}(t)V_{jk}(t)\right)\right] \quad (i\neq j)$$

The above must be tailored as follows:

$$\frac{dV_i(t)}{dt} = b_2 p_i(\beta_i(t)) + 2b_1\left[(H_i(t) - H_{ii}(t))V_i(t) - \sum_{k \neq i} H_{ik}(t)V_{ik}(t)\right] \tag{14}$$

$$\frac{dV_{ij}}{dt} = b_1\left[\begin{array}{l}\left(H_i(t) + H_j(t)\right)V_{ij}(t) - \left(H_{ji}V_i(t) + H_{ij}V_j(t)\right) \\ -\left(\sum_{(k \neq i)} H_{jk}(t)V_{ik}(t) + \sum_{(k \neq j)} H_{ik}(t)V_{jk}(t)\right)\end{array}\right]$$

Now return to specifics; consider (12): for $j \neq k$

$$\frac{d\beta_j(t)}{h(\beta_j(t))} = \frac{d\beta_k(t)}{h(\beta_k(t))}$$

if the bins are filled as suggested. This implies that

$$\beta_j(t) = \frac{b_1 t}{I}, \quad \text{all } j \in (1,2,\ldots,I) \tag{15}$$

and $p_j(\beta_j(t)) = 1/I$.

For the case in which $h(x) = x^{-p}$ it can be seen that

$$H_i(t) = -\frac{p}{b_1 t}, \qquad H_{ij}(t) = -\frac{p}{Ib_1 t} \tag{16}$$

Substitute into (14) to obtain these equations for $V_i(t) \equiv V(t)$, $(\forall i)$; $V_{ij}(t) = W(t)$, $(\forall i \neq j)$:

$$\frac{dV}{dt} = \frac{b_2}{I} - \frac{2p}{t}\left(\frac{I-1}{I}\right)(V - W) \tag{17,a}$$

$$\frac{dW}{dt} = \frac{2p}{It}(V - W), \tag{17,b}$$

from which an equation for $Z(t) = V(t) - W(t)$ emerges:

$$\frac{dZ}{dt} + \frac{2p}{t}Z(t) = \frac{b_2}{I} \tag{18}$$

A solution to (18) over $t \in (L,\infty)$, $L > 0$ is of this form:

$$Z(t) = \frac{b_2 t}{I(1+2p)} + \frac{K(L)}{t^{2p}}, \qquad L \leq t$$

From this and (17,b)

$$\frac{dW(t)}{dt} = \frac{2p}{It} Z(t)$$

we get

$$W(t) = \frac{2p}{1+2p} \frac{b_2}{I^2} t + \frac{K(L)}{2p}\left(\frac{1}{t^{2p}} - \frac{1}{L^{2p}}\right) \qquad L \le t \tag{19}$$

and from (17,a)

$$\frac{dV(t)}{dt} = \frac{b_2}{I} - \frac{2p}{t}\left(\frac{I-1}{I}\right) Z(t)$$

$$V(t) = \frac{b_2 t}{I^2}\left(\frac{2p+I}{2p+1}\right) + K(L)\left(\frac{I-1}{I}\right)\left(\frac{1}{t^{2p+1}} - \frac{1}{L^{2p+1}}\right) \tag{20}$$

Now if $p >> 1$ it is seen that

$$V(t) = W(t) \cong \frac{b_2 t}{I^2} \tag{21}$$

Parenthetically, the comparable figures for independently filled bins are

$$V(t) = \frac{b_2 t}{I} \text{ and } W(t) = 0. \tag{22}$$

From (15) and (20)

$$E[N_i(t)] = \lambda t \frac{b_1}{I} + O(\sqrt{\lambda})$$

$$Var[N_i(t)] = Cov[N_i(t), N_j(t)] = \lambda t \frac{b_2}{I^2} + O(\sqrt{\lambda}) \tag{23}$$

The singular behavior of $W(t)$ and $V(t)$ for small t, as in (19) and (20), can be attributed to the indeterminacy of the bin selection probability, (2), for $N_i(0) = 0$, $\forall_i$, which was the assumed initial condition. The long-time behavior of greedy packing, expressed by (23), is of interest: since $Var[N_i(t)] = Cov[N_i(t), N_j(t)]$ all bin contents are essentially perfectly correlated at any time. Consequently, to order $\sqrt{\lambda}$ the *maximum* bin contents $N_m(t)$, are approximately normal/Gaussian with mean $\lambda t b_1 / I$ and standard deviation $\sqrt{\lambda t b_2} / I$. If bins are filled independently the mean is the same but now $N_m(t)$ is distributed approximately as the maximum of I independent normals, each with standard deviation $\sqrt{\lambda t b_2} / \sqrt{I}$ — considerably larger in a probabilistic sense. Note that putting $p = 0$ in (19), (20) yields the independent result; putting p finite yields other probabilistic options. In the scheduling context the *makespan*, i.e. time to complete all tasks present at time t, is substantially reduced by the current greedy scheme, which is equivalent to what Coffman *et al.* (1991) call *list scheduling* (LS); our approach is *on-line* list scheduling, meaning that tasks are assigned to processors sequentially as they appear in time. It should be pointed out that the moments obtained above can also be derived directly from (1) and (2) by expansion of (5), rather as suggested by Isham [3].

Simulations

Limited informal simulations were conducted in order to check the accuracy of the proposed asymptotic approximations. The simulations were written in APL2 and conducted on an AMDAHL 5995–700A at the Naval Postgraduate School using the LLRANDOM random number generator; cf. Lewis et al. [4]. All simulations were run for time $t = 1$ at the indicated λ-values for two job size distributions, both gamma: the exponential and an extended-tail highly-skewed gamma with shape parameter one-half.

Examination of Table 1 indicates that agreement is good between the asymptotic approximation and simulation results (based on 1000 replications) for the marginal distributional properties of an arbitrary bin when the greedy policy is followed. As anticipated, a considerable reduction in the variance of bin size, and also of upper-tail percentiles, is achieved by greediness, as contrasted to a simple random assignment.

The figures of Table 2, which describe the approximation to the maximum bin size, or makespan in a scheduling context, are serviceable but tend to be low or optimistic, especially for the smaller λ-value of 50. For $\lambda = 300$ the agreement is better and correctly predicts the substantial reduction of mean, variance, and upper percentiles achieved by the greedy policy. Note that numerical agreement between simulation and our asymptotics should improve if the job sizes have smaller variances and third moments.

It can be conjectured, and demonstrated, that a cyclic or round-robin policy of putting every I^{th} arrival in the same bin will tend to reduce within-bin variance and makespan levels. It may be advantageous that both random and round-robin policies require no information concerning current bin size or occupancy at the time an assignment must be made, whereas the greedy policy and others depend on precise distributional forms do require such information. If reduction of bin size or makespan variation is important the acquisition of the information needed to implement a greedy policy may be well worth the cost.

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Table 1. ARBITRARY BIN SIZE

$I = 5$

B: $G = 1$: Gamma, Shape Param. = 1 (Exponential)

$G = 0.5$: Gamma, Shape Param. = 0.5; $E[B] = 1$.

Number Replications = 1000

		Policy								
		Greedy				Random				
				Quantiles				Quantiles		
Mean Demand(λ)	Job Size (B)		Mean	Var	80%	90%	Mean	Var	80%	90%
50	G = 1	App:	10.0	4.0	11.7	12.6	10.0	20.0	13.8	15.7
		Sim:	10.1	5.0	11.9	13.0	10.0	19.6	13.7	16.1
	G = 0.5	App:	10.0	6.0	12.3	13.1	10.0	30.0	14.6	17.0
		Sim:	10.0	8.5	12.1	13.8	10.2	32.6	14.8	18.0
300	G = 1	App:	60.0	24.0	64.1	66.3	60.0	120.0	69.2	74.0
		Sim:	60.2	24.1	64.4	66.4	60.4	112.1	69.5	73.9
	G = 0.5	App:	60.0	36.0	65.0	67.7	60.0	180.0	71.2	77.1
		Sim:	60.2	38.9	65.4	68.0	60.0	172.7	70.7	77.1

Table 2. MAXIMUM BIN SIZE

$I = 5$

B: $G = 1$: Gamma, Shape Param. = 1 (Exponential)

$G = 0.5$: Gamma, Shape Param. = 0.5; $E[B] = 1$.

Number Replications = 1000

		Policy							
		Greedy				Random			
				Quantiles				Quantiles	
Mean Demand(λ)	Job Size (B)	Mean	Var	80%	90%	Mean	Var	80%	90%
50	G = 1	App: 10.0	4.0	11.7	12.6	—	—	17.6	19.1
		Sim: 11.2	5.2	13.1	14.2	15.6	15.1	18.7	23.9
	G = 0.5	App: 10.0	6.0	12.1	13.1	—	—	19.4	21.2
		Sim: 12.2	10.7	14.8	16.5	16.9	28.2	20.7	23.9
300	G = 1	App: 60.0	24.0	64.1	66.3	—	—	78.7	82.3
		Sim: 61.5	24.3	65.6	67.7	73.3	61.5	79.5	83.5
	G = 0.5	App: 60.0	36.0	65.0	67.7	—	—	82.9	87.3
		Sim: 62.3	40.0	67.7	70.3	76.2	107.6	84.3	90.4

— : Not convenient to compute

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